MATH 3060 Assignment 8 solution

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December 2, 2022

1. Let $||\cdot||$ be a norm on $\mathbb{R}^n$. Let $e_1, e_2, \ldots, e_n$ be a basis of $\mathbb{R}^n$ and choose $c > \sup\{||e_i||\}$. For $x = \sum x_i e_i$We have

$$||x|| \leq c \sum |x_i| \leq c\sqrt{n}||x||_2$$

by Cauchy-Schwartz inequality. This in particular shows that the function $||\cdot|| : \mathbb{R}^n \to \mathbb{R}$ is continuous. Let $d = \min\{||x|| : ||x||_2 = 1\}$, then $||x|| \geq d||x||_2$ for any $x \in \mathbb{R}^n$.

2. By Homework 4 Question 1, H is closed. To show H is nowhere dense, let $\epsilon > 0$ and choose $\{x_n\} \in H$. Take $N \in \mathbb{Z}_+$ such that $N > 3/\epsilon$. Define $\{x'_n\}$ by $x'_n = x_n$ for $n \neq N$ and $x'_N = 2/N$. Then $x'_N > 1/N$ but $d_2(\{x_n\}, \{x'_n\}) \leq 3/N < \epsilon$.

3. The set of polynomials is a vector subspace of $C[0,1]$ of countable dimension. By the proof of theorem 4.14 in lecture 23, such subspaces are of 1st category.

4. For $a < b$, define

$$Z_{a,b} = \{f \in M[0,1] : [a,b] \subset \overline{f([0,1])}\}$$

A function $f \in M[0,1]$ has nowhere dense image if and only $f \notin Z_{a,b}$ for any $a, b \in \mathbb{Q}$. So it suffices to show $Z_{a,b}$ is nowhere dense. For $f \notin Z_{a,b}$, then $d = \sup\{d(x, \overline{f([0,1])}) : x \in [a,b]\} > 0$, and for any $f' \in B_d(f)$, we have $f \notin Z_{a,b}$. This shows the complement of $Z_{a,b}$ is open, hence $Z_{a,b}$ is closed.
We next show that $Z_{a,b}$ has empty interior. Pick $f \in Z_{a,b}$ and $\epsilon > 0$. Define g by $g(x) = f(x)$ if $|f(x) - a| > \epsilon/2$ and $g(x) = a + \epsilon/2$ otherwise. Then $||f - g||_\infty < \epsilon$ while $a \notin \overline{f([0,1])}$.